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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) the latest about pi: (2) a series useful in the computation of pi: (3) an ENIAC determination of pi and e to more than 2,000 decimal places: (4) the evolution of extended decimal approximations to pi: and (5) the calculation of pi to 100,265 decimal places. (MP)

REPRINT SERIES

Computation Of π

Edited by William L. Schaaf

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Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

THE COMPUTATION OF π

PREFACE

Although the familiar symbol (π) for pi did not come into general use until a little over two hundred years ago, computing the numerical value of π has engaged the attention of mathematicians from the time of the ancient Egyptions down to the electronic computers of today. Thus the Ahmes Papyrus (Egypt) of about 1800 B.C. gives the area of a circle as

$$\left(d - \frac{d}{9}\right)^2,$$

where d, is the diameter. This is equivalent to taking π as

$$\left(\frac{16}{9}\right)^2,$$

or approximately 3.1604 . . . At about the same time, the Babylonians, the Hindus and the Chinese took π as equal to 3.

The early Greeks were concerned with the problem of squaring the circle, and in the course of his searching, Archimedes, about 250 B.C., assumed the value of π to lie between

$$3\frac{10}{71}\ (= 3.1408\ldots) \text{ and } 3\frac{1}{7}\ (= 3.1428\ldots).$$

The Chinese soon decided (about 100 A.D.) that π was approximately equal to

$$\sqrt{10},$$

or 3.162 . . . About 150 A.D., the renowned Greek astronomer, Ptolemy of Alexandria, using the sexagesimal system of notation, stated that $\pi = 3°8'30''$, or, as we would write it today,

$$3 + \frac{8}{60} + \frac{30}{(60)^2} = 3\frac{17}{120},$$

which gives the approximation 3.1416, or 3.141666 . . . The Hindu mathematician Aryabhata, about 500 A.D., gave two values of π,

$$3\frac{177}{1250} \text{ and } \frac{62{,}832}{20{,}000},$$

both of which give the value 3.1416, exactly. The latter fraction is presumably calculated from the perimeter of an inscribed polygon of 384 sides.

For the next thousand years or more mathematicians in many lands struggled with the problem, but with little progress. Finally, about 1580, Francois Vieta, a pioneer French algebraist, using a polygon of 393,216 sides, found π correct to nine decimal places, placing it between 3.1415926535 and 3.1415926537. It appears that he was the first mathematician to use an infinite product in this connection, asserting that

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

By way of contrast, contemporary mathematicians have computed the value of π to more than 100,000 decimal places. This is a fantastic achievement from any point of view. The story of this long evolution lasting nearly 4000 years is indeed fascinating; the climax is vividly set forth in the present collection of essays.

William L. Schaaf

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AMERICAN MATHEMATICAL MONTHLY:

J. S. FRAME. "*A Series Useful in the Computation of π*", vol. 42 (1935), p. 499-501.

THE MATHEMATICS TEACHER:

HOWARD EVES. "*The Latest About π*", vol. 55 (Feb. 1962), p. 129-130.

J. W. WRENCH, JR., "*The Evolution of Extended Decimal Approximations to π*", vol. 53 (Dec. 1960), p. 644-650.

MATHEMATICAL TABLES AND OTHER AIDS TO COMPUTATION:

GEORGE W. REITWEISNER. "*An Eniac Determination of π and e to More Than 2000 Decimal Places*", vol. 4 (1950), p. 11-15.

RECREATIONAL MATHEMATICS MAGAZINE:

"*Did You Know That π Has Been Calculated to 100,265 Decimal Places?*" *No. 8, April 1962, pp. 20-21.*

FOREWORD

The numerical value of π can be approximated by either of two general methods with as close an approximation to its "true" value as we wish. One method is geometrical. This is the classical approach first used by the Greek geometers and by mathematicians generally until comparatively modern times, that is until about 1650. It involves computing the perimeters of polygons inscribed in and circumscribed about a circle, and assuming that the circumference is intermediate between these perimeters. As the number of sides of the polygons is increased, the approximation becomes more accurate. In fact, if the areas of the polygons are used instead of the perimeters, an even better approximation can be obtained.

The second method, the modern approach, depends upon an expansion of π into some equivalent analytical expression such as a converging infinite series or a convergent infinite product. One of the first mathematicians to use such an expression was Vieta, as we have already seen. Another was John Wallis, who showed, in 1656, that π could be expanded into the infinite product:

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\dots$$

Many other mathematicians have developed various expansions for evaluating π, among them James Gregory, G. W. von Leibniz, John Machin, Leonard Euler, and C. F. Gauss.

Perhaps a few words of explanation about infinite series will help you to understand the following articles better.

A succession of numbers which follows a definite law or pattern is called a *finite sequence;* for example,

$$\text{(a) } 2, 4, 8, \dots 2^n,$$

or

$$\text{(b) } 1, \frac{1}{4}, \frac{1}{9}, \dots \frac{1}{2^n},$$

where n is a positive integer.

A sequence that is endless, having a first term but no last term, is called an *infinite sequence.*

If we consider the sum of the first n terms of a finite sequence, we refer to the indicated sum as a *finite series*; thus

$$\sum_{n=1}^{n=6}(n^2)=1+4+9+16+25+36=91.$$

We designate the indicated sum of the terms of an infinite sequence as an *infinite series.* But this is not a sum in the usual sense of the word, because the terms of an infinite series can never all be added term by term.

If the succession of partial sums of an infinite series increases indefinitely as n increases indefinitely, the series is said to be *divergent*, and the "sum" of the series is meaningless.

If, on the other hand, the succession of partial sums of an infinite series approaches a limiting value as n increases indefinitely, the series is said to be *convergent*, and the "sum" of the series refers to this limiting value. For example, in the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots,$$

the limiting value is 2. The sum of any finite number of terms of this series, however great, is always *less* than 2; but by taking more and more terms, the partial sum can be made as close to 2 as we wish. The limiting value "2" is called the sum of the convergent infinite series in question.

The Latest About π

by *Howard Eves*

On July 29, 1961, Dr. Daniel Shanks and Dr. John W. Wrench, Jr. computed π to 100,265 decimal places on an IBM 7090 system in the IBM Data-center in New York. The computation took 8 hours 43 minutes, including 42 minutes to convert the final result from binary to decimal form. A check run, using a second formula, confirmed the accuracy of the first run to 70,695 decimals, and subsequent runs on 7090 computers in the Washington area showed that a machine error occurred in the initial run. Dr. Shanks and Dr. Wrench now have results that agree perfectly (including the conversion and printing) to 333,075 bits or 100,265 decimal places.

The first computation employed the formula

$$\pi = 24 \tan^{-1}(\tfrac{1}{8}) + 8 \tan^{-1}(\tfrac{1}{57}) + 4 \tan^{-1}(\tfrac{1}{239}),$$

which was published by Carl Störmer in 1896. This formula is especially well adapted to binary computers, inasmuch as the evaluation of powers of $\frac{1}{8}$ on such computers can be accomplished simply by shifting.

The check computation was based on the formula

$$\pi = 48 \tan^{-1}(\tfrac{1}{18}) + 32 \tan^{-1}(\tfrac{1}{57}) - 20 \tan^{-1}(\tfrac{1}{239}),$$

of Gauss, which was used by George Felton to compute π to 10,021 decimal places on a Pegasus in 1958. Because of the overlapping terms in the two formulas used, the check consisted almost entirely of the computation of $48 \tan^{-1}(\frac{1}{18})$. This required 4 hours 22 minutes on the IBM 7090 system.

Following the discovery of a machine error in the evaluation of $24 \tan^{-1}(\frac{1}{8})$beyond the 70,695 decimal place, all the arctangents were individually recomputed on a second 7090 system, and complete agreement was reached in all phases of the calculation to 100,265 places.

On September 11, 1961, the 7090 system prepared a count of the frequency distribution of the decimal digits of π in successive chiliads. Comparison with this latest count, carried to 100,000 places, revealed a few errors in Dr. Wrench's enumeration of the distribution of the digits 7, 8, 9 as published in Table 1 of his paper, "The evolution of extended decimal approximations to π," in THE MATHEMATICS TEACHER, LIII (Dec., 1960), 648. This earlier count had been based on a computation of π to 16,167 decimal pleces obtained on July 20, 1959, using a program of Francois Genuys, on an IBM 704 system at the Commissariat a l'Energie Atomique in Paris. In Table 1 of Dr. Wrench's article in THE MATHEMATICS TEACHER, the last four entries in the 7-column should read 1258, 1342, 1439, and

1546, respectively. In the 8-column read 1243, 1336, 1455, and 1543, and in the 9-column read 1306, 1418, 1513, 1615. With these corrections, Table 1 is entirely free from errors.

Furthermore, on August 22, 1961, Dr. Shanks and Dr. Wrench also computed e to 100,265 decimal places on an IBM 7090 system in 2 hours 25 minutes, exclusive of the conversion to decimal form, which again required 42 minutes. The well-known factorial series was used, and a total of 25,266 reciprocal factorials were evaluated to the stated accuracy. This confirms the 60,000 decimal place computation of e on the Illiac by D. J. Wheeler in December, 1952. Wheeler's calculation required 40 hours on the Illinois computer. The accuracy of the 100,265 decimal places constituting this latest approximation to e has been confirmed by a second calculation, which gave the respective sums of the even- and odd-numbered terms of the factorial series, yielding approximations to both e and $1/e$ to this accuracy.

Dr. Shanks and Dr. Wrench have prepared a joint paper on their calculations of π, which appears in the January, 1962, issue of *Mathematics of Computation*. Appended to their paper is the value of π truncated to 100,000 decimal places.

Foreword

As it turns out, some infinite series converge more rapidly than others. Consider, for example, the series

$$\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \tag{1}$$

If we set $x = 1$, we get Gregory's series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots; \tag{2}$$

but this series converges too slowly for purposes of computation.

On the other hand, Machin's formula,

$$\frac{\pi}{4} = 4 \text{ arc tan } \frac{1}{5} - \text{arc tan } \frac{1}{239} \tag{3}$$

used in conjunction with the expansion (1) above, converges much more rapidly. In fact, you can get a rather good approximation to π simply by taking the first four terms of (1) when $x = 1/5$ (or .2), together with first term of (1) when $x = 1/239$. Try it and see for yourself!

The reader who is familiar with trigonometry, may be interested in the derivation of Machin's formula. To prove that

$$\frac{\pi}{4} = 4 \text{ arc tan } \left(\frac{1}{5}\right) - \text{arc tan } \left(\frac{1}{239}\right),$$

let arc tan $1/5 = \alpha$, so that $\tan \alpha = 1/5$. (1)

$$\text{Then } \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12}. \tag{2}$$

$$\text{and } \tan 4\alpha = \frac{\frac{10}{12}}{1 - \frac{25}{144}} = \frac{120}{119}. \tag{3}$$

Since $\tan 4\alpha$ is very nearly equal to 1, we see that 4α is approximately equal to $\pi/4$. Now let $4\alpha = \pi/4 + \text{arc tan } x$. (4)

$$\text{Recall that } \tan\left(A + \frac{\pi}{4}\right) = \frac{\tan A + 1}{1 - \tan A} = \frac{1 + \tan A}{1 - \tan A}. \tag{5}$$

$$\text{Hence, } \frac{120}{119} = \tan 4\alpha = \tan\left(\text{arc tan } x + \frac{\pi}{4}\right) = \frac{1 + x}{1 - x}, \text{ and } x = \frac{1}{239}. \tag{6}$$

Therefore, from (4), we have $\frac{\pi}{4} = 4\alpha - \text{arc tan } x$,

or $$\frac{\pi}{4} = 4 \text{ arc tan } \left(\frac{1}{5}\right) - \text{arc tan } \left(\frac{1}{239}\right).$$

A Series Useful in the Computation of π

by *J. S. Frame*

One of the standards ways of computing π is based on Machin's formula:

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \tag{1}$$

and the series expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \tag{2}$$

W. Shanks used precisely this in computing π to 707 decimal places. In applying this series to the case $x = 1/5$, the individual terms are easily computed as decimals, and the series converges rapidly enough so that 35 terms suffice for 50-place accuracy. When we set $x = 1/239$, however, the individual terms, involving powers of 1/239, are not easily expressed as decimals, so that computation beyond 15 decimals is laborious despite the rapid convergence. If, however, we expand the terms in powers of 1/240, we obtain a new series which converges rapidly, and whose terms are easier to compute as decimals. The result is expressed by the formula:

$$\begin{aligned} \tan^{-1} = \frac{t}{1-t} &= \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{4}\right)\frac{(t\sqrt{2})^n}{n} \\ &= \frac{t}{1} + \frac{2t^2}{2} + \frac{2t^3}{3} + 0 - \frac{4t^5}{5} - \frac{8t^6}{6} - \frac{8t^7}{7} - 0 + \dots . \end{aligned} \tag{3}$$

The terms are alternately positive and negative in groups of three, so the error in breaking off the series is less in absolute value than the first group omitted. The series converges for $|t| < 1/\sqrt{2}$. Setting $t = 1/240$, we obtain the series

$$\begin{aligned} \tan^{-1}\frac{1}{239} = \frac{1}{240} &+ \frac{2}{2}\left(\frac{1}{240}\right)^2 \quad \frac{2}{3}\left(\frac{1}{240}\right)^3 + 0 \\ &- \frac{4}{5}\left(\frac{1}{240}\right)^5 - \frac{8}{6}\left(\frac{1}{240}\right)^6 - \frac{8}{7}\left(\frac{1}{240}\right)^7 + \dots . \end{aligned}$$

The computation is conveniently arranged as follows: Divide 1 by 240, this by 120, this in turn by 240, and so on alternately. This takes care of the numerators automatically. It remains only to divide each term by the corresponding exponent, and add and subtract appropriate terms. Sixteen terms give 50-place accuracy.

The proof of formula (3) is a special case of the following: Let

$$x = \frac{at+b}{ct+d}; \quad z = re^{i\theta} = \frac{ia-c}{-ib+d}.$$

Then

$$2i \tan^{-1} x = \log \frac{1+ix}{1-ix} = \log \frac{(ct+d)+i(at+b)}{(ct+d)-i(at+b)}$$

$$= \log \frac{(c+ia)t+(d+ib)}{(c-ia)t+(d-ib)} = \log \frac{1+\dfrac{c+ia}{d+ib}t}{1+\dfrac{c-ia}{d-ib}t} + \log \frac{d+ib}{d-ib},$$

$$\tan^{-1}\frac{at+b}{ct+d} - \tan^{-1}\frac{b}{d} = \frac{1}{2i}\log\frac{1-\bar{z}t}{1-zt} = \sum_{n=1}^{\infty}\frac{z^n-\bar{z}^n}{2i}\frac{t^n}{n} = \sum_{n=1}^{\infty}(r^n \sin n\theta)\frac{t^n}{n}.$$

This series converges for $|t| < 1/r$, but it is useful for computation only when the values of $r^n \sin n\theta$ are convenient rational quantities. If $z = i$, we have $x = t$, and obtain the series (2). The other case of interest is $z = 1 + i$, $x = t/(1-t)$, which leads to formula (3), and can be applied to the computation of π as discussed above. This same series (3) can be used to advantage in computing $\tan^{-1} 1/239$ by means of the formula

$$\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{41} - 2\tan^{-1}\frac{1}{99}. \tag{4}$$

Foreword

One of the first large electronic computers ever built, the ENIAC was designed and constructed at the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, in 1946. The name stands for "*Electronic Numerical Integrator and Calculator*". It was capable of performing 5000 additions per second and up to 500 multiplications per second.

Advances and improvements in electronic computers have been unbelievingly rapid in the twenty odd years since ENIAC first appeared. Today's (1965) machines can perform 100,000 additions per second and 10,000 multiplications per second.

An ENIAC Determination of π and e to more than 2000 Decimal Places

George W. Reitwiesner

Early in June, 1949, Professor John von Neumann expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits, suggesting the employment of one of the formulas:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$
$$\pi/4 = 8 \arctan 1/10 - 4 \arctan 1/515 - \arctan 1/239$$
$$\pi/4 = 3 \arctan 1/4 + \arctan 1/20 + \arctan 1/1985$$

in conjunction with the Gregory series

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1} .$$

Further interest in the project on π was expressed in July by Dr. Nicholas Metropolis who offered suggestions about programming the calculation.

Since the possiblity of official time was too remote for consideration, permission was obtained to execute these projects during two summer holiday week ends when the ENIAC would otherwise stand idle, and the planning and programming of the projects was undertaken on an extra-curricular basis by the author.

The computation of e was completed over the July 4th week end as a practice job to gain experience and technique for the more difficult and longer project on π. The reciprocal factorial series was employed:

$$e = \sum_{n=0}^{\infty} (n!)^{-1} .$$

The first of the above-mentioned formulas was employed for the computation of π; its advantage over the others will be explained later. The computation of π was completed over the Labor-Day week end through the combined efforts of four members of the ENIAC staff: Clyde V. Hauff (who checked the programming for π), Miss Homé S. McAllister (who checked the programming for e), W. Barkley Fritz and the author, taking turns on eight-hour shifts to keep the ENIAC operating continuously throughout the week end.

While the programming for e is valid for a little over 2500 decimal places and, with minor alterations, can be extended to much greater range, and while the programming for π is valid for around 7000 decimal places, the arbitrarily selected limit of 2000+ was a convenient stopping point for e and about all that could be anticipated for a week end's operation for π.

While the details of the programming for each project were completely different, the general pattern of procedure was roughly the same, and both projects will be discussed together. In both projects the ENIAC'S divider was employed to determine a chosen number i of digits of each successive term of the series being computed, the remainder after each division being stored in the ENIAC'S memory and the digits of each term being added to (or subtracted from) the cumulative total. After performing this operation for as many successive terms as practicable, the remainders for these terms were printed on an I.B.M. card (the standard input-output vehicle for the ENIAC), and the process was repeated, continuing through some term beyond which the digits of and remainders for all further terms would be zeros. At this point was printed the cumulative total of the digits of the individual terms, which yielded (after adjustment for carry-over) the actual digits of the series being determined.

The cards bearing the remainders then were fed into the ENIAC reader, and the entire process was repeated for the next i digits, the ENIAC reading each remainder in turn and placing it before the digits of the appropriate term. Each deck of cards bearing remainders was then employed to determine the "next" i digits and the "next" deck of "remainder" cards continuing through the first stopping point beyond the 2000th decimal place. The cards bearing the cumulative totals of sets of i digits of the terms were then adjusted for carry-over into each preceding set of i digits. In the case of e this yielded the final result; in the case of π all the above described operations were performed once for each inverse tangent series, so that each set of "cumulative total" cards, adjusted for carry-over, yielded the digits of one of the series, the final result being determined by the combination of these series in appropriate manner.

The number of places i chosen for each interval of computation, the maximum magnitude of each remainder, the amount of memory space available, and the details of divider operation (the number of places to which division can be performed to yield a positive remainder, and the necessary conditions of relative and absolute positioning of numerator and denominator) all were interrelated, and where opportunity for selection existed, that selection was made which provided maximum efficiency of computation. In the case of π there was imposed the additional requirement that identical programming apply for all series employed, and for this reason the formula:

$$\pi/4 = 4 \operatorname{arctan} 1/5 - \operatorname{arctan} 1/239$$

was superior to the other two.

In order to insure absolute digital accuracy, the programming was arranged so that one half applied to computation and the other half to checking. Before any deck of "remainder" cards was employed to determine the next i digits, the cards were reversed and employed in the checking sequence to confirm each division by a multiplication and each addition by a subtraction and vice versa, reproducing the previous deck of "remainder" cards and insuring that the cumulative total reduced to zero. (In the case of e this was a simple inversion of the computation; in the case of π the factor $(2n + 1)^{-1}$ in each term made it a more complicated affair). After the correctness of each deck was established through this checking, the "remainder" cards were rereversed, and the computation proceeded for the next i digits.

Since the determination of each i digits was not begun until the determination of the previous i digits had been confirmed by checking, the ENIAC stood idle during the reversals and rereversals and comparisons of the decks in the computation of e; in the case of π, however, the ENIAC was never idle, for operation on each series was alternated with operation on the other, card-handling on either being accomplished while the other was being operated upon by the ENIAC. In the case of e, insurance against any undiscovered accidental misalignment of cards was provided by rerunning the entire computation without checking, i.e., without card reversals, confirming the original results; in the case of π, the same assurance was provided by a programmed check upon the identification numbers of each successive card in both computation and checking.

In the case of e, there was printed (in addition to each "remainder" card) a card containing the current i digits of $(n!)^{-1}$ for $n = 20K$; $K = 1, 2, 3 \ldots$; in the case of π only remainder and final total cards were printed.

The ENIAC determinations of both π and e confirm the 808—place determination of e published in *MTAC*, v. 2, 1946, p. 69, and the 808—place determination of π published in *MTAC*, v. 2, 1947, p. 245, as corrected in *MTAC*, v. 3, 1948, p. 18-19.

Only the following minor observation is offered at this time concerning the randomness of the distribution of the digits. Publication on this subject will, however, be forthcoming soon. A preliminary investigation has indicated that the digits of e deviate significantly from randomness (in the sense of staying closer to their expectation values than a random sequence of this length normally would) while for π no significant deviations have so far been detected.

The programming was checked and the first few hundred decimal places of each constant were determined on a Sunday before each holiday week end mentioned above, the principal effort being made on the longer week end. The actual required machine running time for both computation and checking in the case of e was around 11 hours, though card-handling time approximately doubled this, and the recomputation without checking added about 6 hours more; actual required machine running time (including card-handling time) for π was around 70 hours.

The following values of π and e have been rounded off to 2035D and 2010D respectively.

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
82148 08651 32823 06647 09384 46095 50582 23172 53594 08128
48111 74502 84102 70193 85211 05559 64462 29489 54930 38196
44288 10975 66593 34461 28475 64823 37867 83165 27120 19091
45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436

78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
94657	64078	95126	94683	98352	59570	98258			

e = 2.71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
15738	34187	93070	21540	89149	93488	41675	09244	76146	06680
82264	80016	84774	11853	74234	54424	37107	53907	77449	92069
55170	27618	38606	26133	13845	83000	75204	49338	26560	29760
67371	13200	70932	87091	27443	74704	72306	96977	20931	01416
92836	81902	55151	08657	46377	21112	52389	78442	50569	53696
77078	54499	69967	94686	44549	05987	93163	68892	30098	79312
77361	78215	42499	92295	76351	48220	82698	95193	66803	31825
28869	39849	64651	05820	93923	98294	88793	32036	25094	43117
30123	81970	68416	14039	70198	37679	32068	32823	76464	80429
53118	02328	78250	98194	55815	30175	67173	61332	06981	12509
96181	88159	30416	90351	59888	85193	45807	27386	67385	89422
87922	84998	92086	80582	57492	79610	48419	84443	63463	24496
84875	60233	62482	70419	78623	20900	21609	90235	30436	99418
49146	31409	34317	38143	64054	62531	52096	18369	08887	07016
76839	64243	78140	59271	45635	49061	30310	72085	10383	75051
01157	47704	17189	86106	87396	96552	12671	54688	95703	50354
02123	40784	98193	34321	06817	01210	05627	88023	51930	33224

74501	58539	04730	41995	77770	93503	66041	69973	29725	08868
76966	40355	57071	62268	44716	25607	98826	51787	13419	51246
65201	03059	21236	67719	43252	78675	39855	89448	96970	96409
75459	18569	56380	23637	01621	12047	74272	28364	89613	42251
64450	78182	44235	29486	36372	14174	02388	93441	24796	35743
70263	75529	44483	37998	01612	54922	78509	25778	25620	92622
64832	62779	33386	56648	16277	25164	01910	59004	91644	99828
93150	56604	72580	27786	31864	15519	56532	44258	69829	46959
30801	91529	87211	72556	34754	63964	47910	14590	40905	86298
49679	12874	06870	50489	58586	71747	98546	67757	57320	56812
88459	20541	33405	39220	00113	78630	09455	60688	16674	00169
84205	58040	33637	95376	45203	04024	32256	61352	78369	51177
88386	38744	39662	53224	98506	54995	88623	42818	99707	73327
61717	83928	03494	65014	34558	89707	19425	86398	77275	47109
62953	74152	11151	36835	06275	26023	26484	72870	39207	64310
05958	41166	12054	52970	30236	47254	92966	69381	15137	32275
36450	98889	03136	02057	24817	65851	18063	03644	28123	14965
50704	75102	54465	01172	72115	55194	86685	08003	68532	28183
15219	60037	35625	27944	95158	28418	82947	87610	85263	98139
55990	06738								

Values of the auxiliary numbers arccot 5 and arccot 239 to 2035D are in the possession of the author and also have been deposited in the library of Brown University and the UMT File[1] of *MTAC*.

[1]See *MTAC*, v. 4, p. 29.

Foreword

Significantly, the real numbers of elementary algebra fall into two disjoint sets: (1) the rational numbers, and (2) the irrational numbers. A *rational number* is a number that can be expressed as the ratio of two integers, as, for example, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{6}{1}$, etc. Every rational number when expressed in decimal fraction form yields either a terminating decimal or a repeating decimal. Thus,

$$\frac{7}{8} = .875, \text{ and } \frac{2}{11} = .181818 \ldots .$$

It is not difficult to show that between any two rational numbers there exist infinitely many other rational numbers. Thus, if the rational numbers were associated with points on a line, it would seem as if the line were "completely filled" with points.

Although it is difficult to picture it, such a line is *not* completely filled with points. Strangely enough, it is full of "holes", that is, there are many points which have no rational numbers assigned to them. The numbers that "belong" to these points are called irrational numbers.

An *irrational number* is a number that is not rational, that is, it cannot be expressed as the quotient of two integers. The existence of non-rational numbers is easily shown. A classical proof was given by Pythagoras over 2000 years ago, as follows. Assume that $\sqrt{2}$ is rational. Let $\sqrt{2} = a/b$, where a and b are relatively prime. Then

$$2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$

$$\text{or} \quad 2b^2 = a^2. \tag{1}$$

Hence a^2 is an even number; therefore a is also an even number. But if a is an even number, it can be expressed as $2k$, where k is any positive integer. Thus

$$2b^2 = a^2 = (2k)^2 = 4k^2,$$

$$\text{or} \quad b^2 = 2k^2.$$

Hence b^2 is an even number, and therefore b is an even number. Since both a and b have been proved to be even numbers, the assumption that a and b are relatively prime is false, and so the assumption that $\sqrt{2} = a/b$ is false. In short, $\sqrt{2}$ cannot be expressed as the ratio of two integers. The proof can be generalized to $\sqrt{N}$, where N is any integer which is not the square of another integer.

One of the reasons for the many attempts to find the value of π to so many decimal places is the desire to learn something about the *distribution of the digits* in the extended approximation of π. It has been proved that π is an irrational number, that is a number which when expressed as a decimal in base 10, yields a non-terminating, non-repeating decimal.

An irrational number is said to be a *normal number* if all the digits occur with equal frequency, and if all blocks of digits of the same length occur with equal frequency. From the standpoint of the theory of numbers and higher analysis, mathematicians are curious about the distribution of the digits in the numerical approximation of π. It is believed that π is a normal number with respect to base 10, but it is not yet known whether π is normal to any base. These and related questions are of considerable interest to modern mathematicians.

The Evolution of Extended Decimal Approximations to π

by J. W. WRENCH, JR.,

In his historical survey of the classic problem of "squaring the circle," Professor E. W. Hobson [1]* distinguished three distinct periods, characterized by fundamental differences in method, immediate aims, and available mathematical tools.

The first period—the so-called geometrical period—extended from the earliest empirical determinations of the ratio of the circumference of a circle to its diameter to the invention of the calculus about the middle of the seventeenth century. The main effort was directed toward the approximation of this ratio by the calculation of perimeters or areas of regular inscribed and circumscribed polygons.

The second period began in the middle of the seventeenth century and lasted for more than a hundred years. During this period the methods of the calculus were employed in the development of analytical expressions for π in the form of infinite series, products, and continued fractions.

The third period, which extended from the middle of the eighteenth century to nearly the end of the nineteenth century, was devoted to studies of the nature of the number π. J. H. Lambert [2] proved the irrationality of π in 1761, and F. Lindemann [3] first established its transcendence in 1882.

This article is concerned with the second period and its sequel, which extends to the present day.

According to Hobson [1], the first analytical expression discovered in this period is the infinite product

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$$

which was published by John Wallis [4] in 1655.

Lord Brouncker, the first president of the Royal Society, about 1658 found the infinite continued fraction

$$\frac{\pi}{4}=\frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots,$$

which was shown subsequently by Euler to be equivalent to the alternating series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots,$$

known to G. W. Leibniz in 1674.

*Numbers in brackets refer to the references listed at the end of the article.

The great majority of calculations of π to many decimal places have been based upon the power series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, -1 \leq x \leq 1,$$

which was discovered in 1671 by James Gregory [5]. He failed, however, to note explicitly the special case corresponding to $x = 1$, which is ascribed to Leibniz.

Sir Isaac Newton [6] in 1676 discovered the power series

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \dots,$$

$$-1 \leq x \leq 1,$$

which has been used by a few computers of π.

In 1755 Leonhard Euler [7] obtained the following useful series:

$$\arctan x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \dots\right\}.$$

It was by means of Gregory's series, taking $x = 1/\sqrt{3}$, that Abraham Sharp [8], at the suggestion of the English astronomer Edmund Halley, computed π to 72 decimal places in 1699, thereby nearly doubling the greatest accuracy (39 decimal places) attained by earlier computers, who had used geometrical methods. Sharp's calculation was extended by Fautet de Lagny [9] in 1719 to 127 decimals (the 113th place has a unit error).

Newton set $x = -\frac{1}{2}$ in his series, and thereby computed π to 14 places. A Japanese computer, Matsunaga Ryohitsu [10], used the same procedure to evaluate π correct to 49 decimal places in 1739. About 1800 a Chinese, Chu Hung, calculated π to 40 places (25 correct) by this series [10].

Most computers of π in modern times have used Gregory's series in conjunction with certain arctangent relations. Only nine of these relations have been employed to any extent in such computations. We shall now consider these formulas, arranged according to the increasing precision of the approximations computed by their use.

$$\text{I. } \frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$$

Euler [7] in 1755 used this relation in conjunction with his series for arctan x to compute π correct to 20 decimal places in one hour. Baron Georg von Vega [11] in 1794 employed Gregory's series and the preceding relation to evaluate π to 140 decimal places, of which the first 136 were correct. This precision was exceeded by that attained by an unknown calculator whose manuscript, containing an approximation correct to 152 places, was seen in the Radcliffe Library at Oxford toward the close of the eighteenth century.

$$\text{II. } \frac{\pi}{4} = 4 \arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$$

Euler published this relation in 1764. It was used by William Rutherford [12] in 1841 to compute π to 208 places (152 correct).

$$\text{III. } \frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$$

This formula was supplied the calculating prodigy Zacharias Dahse [13] by L. K. Schulz von Strassnitzky of Vienna. Within a period of two months in 1844, Dahse thereby evaluated π correct to 200 places.

$$\text{IV. } \frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

First published by Charles Hutton [14] in 1776, this relation was used by W. Lehmann [15] of Potsdam to compute π to 261 decimals in 1853. Tseng Chi-hung [16] in 1877 used the same formula to evaluate π to 100 decimals in a little more than a month.

$$\text{V. } \frac{\pi}{4} = 2 \arctan\frac{1}{3} + \arctan\frac{1}{7}$$

The relation was also published by Hutton [14] in 1776, and independently by Euler in 1779. Vega [17] used it in 1789 to compute 143 decimals (126 correct). In order to remove the uncertainty caused by the discrepant approximations of Rutherford and Dahse, Thomas Clausen [18] extended the calculation to 248 correct decimals in 1847, and Lehmann [15] reached 261 decimals in 1853 by this formula, confirming his independent calculation of π to the same extent by relation IV. Edgar Frisby [19] in Washington, D. C. used relation V in conjunction with Euler's series to compute π to 30 places in 1872.

$$\text{VI. } \frac{\pi}{4} = 3 \arctan\frac{1}{4} + \arctan\frac{1}{20} + \arctan\frac{1}{1985}$$

This formula was published by S. L. Loney [20] in 1893, by Carl Störmer [21] in 1896, and was rediscovered by R. W. Morris [22] in 1944. By means of this formula D. F. Ferguson, then of the Royal Naval College, Eaton, Chester, England, performed a longhand calculation of π to 530 decimal places between May 1944 and May 1945. At that time he discovered a discrepancy between his approximation and the final result of William Shanks – discussed under formula IX – beginning with the 528th place. The first notice of an error in Shank's well-known approximation appeared in a note [22] published by Ferguson in March 1946. He continued his calculation of π and in July 1946 published [23] a correction to

Shank's value through the 620th decimal place. Subsequently, Ferguson used a desk calculator to reach 710 decimals [24] by January 1947, and finally 808 decimals [25] by September 1947.

$$\text{VII. } \frac{\pi}{4} = 8 \arctan \frac{1}{10} - \arctan \frac{1}{239} - 4 \arctan \frac{1}{515}$$

S. Klingenstierna discovered this relation in 1730; it was rediscovered more than a century later by Schellbach [26]. It was used by C. C. Camp [27] in 1926 to evaluate $\pi/4$ to 56 places. D. H. Lehmer [28] recommended it in conjunction with the next formula for the calculation of π to many figures. G. E. Felton on March 31, 1957 completed a calculation of π to 10021 places on a Pegasus computer at the Ferranti Computer Centre in London. This required 33 hours of computer time. The result was published to 10000 places [29]. A check calculation using formula VIII revealed that, because of a machine error, this result was incorrect after 7480 decimal places.

Gauss [30] investigated the derivation of arctangent relations and reduced it to a problem in Diophantine analysis. Relation VIII is one of several formulas he developed. J. P. Ballantine [31] substantiated Lehmer's claim that this formula is especially effective for extensive calculation, by discussing its use in conjunction with Euler's series for the arctangent.

$$\text{VIII. } \frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

Felton carried out a second calculation to 10021 places, and by March 1, 1958 had removed all discrepancies from his results, so that the approximations computed from formulas VII and VIII agreed to within 3 units in the 10021st decimal place. The corrected result remains unpublished.

$$\text{IX. } \frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

This is the most celebrated of all the relations of this kind. John Machin, its discoverer, computed π correct to 100 decimals by means of it in conjunction with Gregory's series, and the result [32] appeared in 1706. Clausen [18] in 1847 used this relation in addition to Hutton's formula V to compute π to 248 decimal places, as has already been noted.

Rutherford resumed his calculation of π in 1852, using Machin's formula this time, as did his former pupil William Shanks. Shanks's first published approximation to π contained 530 decimal places, and was incorporated in Rutherford's note [33], published in 1853, which set forth his approximation to 441 decimals. Later that year Shanks published his book [34] containing an approximation to 607 places and giving all details of the calculation to 530 places. It is now known that Shanks's value was incorrectly calculated beyond 527 decimal places. The accuracy of that value was further vitiated by a blunder committed by Shanks in correcting his copy prior to publication, with the result that similar errors appear in decimal places 460–462 and 513–515. These errors persist in Shanks's first paper of 1873 [35] containing the extension to 707 decimals of his earlier approximation. His second paper of that year [36] which contained his final approximation to π, gives corrections of these errors; however, there appears an inadvertent typographical error in the 326th decimal place of his final value. In retrospect, we now realize that Shanks's first value published in 1853 was the most accurate he ever published.

The accuracy of Shanks's approximation to at least 500 decimals was confirmed by the independent calculations of Professor Richter [37] of Elbing, Germany, who in 1853–1854 computed successive approximations to 330, 400, and 500 places. Richter's communications do not reveal the formula that he used.

Machin's formula was used by H. S. Uhler in an unpublished computation correct to 282 places, which was completed in August 1900.

F. J. Duarte computed π correct to 200 places by this method in 1902. The result was published [38] six years later.

As a by-product of his calculation of the natural logarithms of small primes, Uhler in 1940 noted [39] confirmation to 333 decimal places of Shanks's approximation.

In December 1945, Professor R. C. Archibald suggested that the writer undertake the computation of π by Machin's formula in order to provide an independent check of the accuracy of Ferguson's calculations. With the collaboration of Levi B. Smith, who evaluated arctan 1/239 to 820 decimal places, the writer computed π to 818 places by February 1947, using a desk calculator. The result was published [24] to 808 places in April 1947, and was verified to 710 places by Ferguson in a note published concurrently [24]. The limit of 808 decimals in the published value was chosen to provide precision comparable to that obtained by P. Pedersen [40] in his approximation to e.

Collation of this 808-place approximation with results obtained by Ferguson later that year revealed several erroneous figures beyond the 723rd place in the writer's approximation to arctan $1/5$. These errors vitiated the corresponding figures in the approximation to π. Corrections of these errors and extensions of Ferguson's results appeared in a joint paper [25] by Ferguson and the writer in January 1948, which concluded with an 808-place approximation to π of guaranteed accuracy.

Subsequently, Smith and the writer resumed their calculations and by June 1949 had obtained an approximation to about 1120 decimal places [41]. Before final checking of this extension could be completed, the ENIAC (Electronic Numerical Integrator and Computer) at the Ballistic Research Laboratories, Aberdeen Proving Ground, was employed by George W. Reitwiesner and his associates in September 1949 to evaluate π to about 2037 places (2040 working decimals) in a total time (including card handling) of 70 hours [42]. Machin's formula was also used in this computation.

In November 1954, Smith and the writer extended their calculation to 1150 places, and in January 1956 reverted to this work once more to attain their final result, which was terminated at 1160 places, of which the first 1157 agree with those obtained on the ENIAC.

A calculation of π was performed in duplicate on the NORC (Naval Ordnance Research Calculator) in November 1954 and in January 1955 as a demonstration problem, prior to the delivery of that computer to the U. S. Naval Proving Grounds at Dahlgren, Virginia. Again, Machin's formula was selected, and the calculation was completed to 3093 decimal places in 13 minutes running time. A report of this work, in which the value of π was presented unrounded to 3089 decimal places, was published by S. C. Nicholson and J. Jeenel [43] of the Watson Scientific Computing Laboratory, in New York.

In January 1958, Francois Genuys [44] programmed and carried out the evaluation of π correct to 10000 decimal places on an IBM 704 Electronic Data Processing System at the Paris Data Processing System at the Paris Data Processing Center. Machin's formula in conjunction with Gregory's series was used. Only 40 seconds were required to attain the 707 decimal-place precision reached by Shanks, and one hour and forty minutes was required to reach the 10000 places of the final result.

On July 20, 1959, the program of Genuys was used on an IBM 704 system at the Commissariat a l'Energie Atomique in Paris to compute π to 16167 decimal places. This latest approximation is unpublished at present.

TABLE 1

CUMULATIVE DISTRIBUTION OF THE FIRST 16000 DECIMAL DIGITS OF π

THOUSAND	DIGIT									
	0	1	2	3	4	5	6	7	8	9
1	93	116	103	102	93	97	94	95	101	106
2	182	212	207	188	195	205	200	197	202	212
3	259	309	303	265	318	315	302	287	310	332
4	362	429	408	368	405	417	398	377	405	431
5	466	532	496	459	508	525	513	488	492	512
6	557	626	594	572	613	622	619	606	582	609
7	657	733	692	686	702	730	708	694	680	718
8	754	833	811	781	809	834	816	786	764	812
9	855	936	911	884	910	933	914	883	854	920
10	968	1026	1021	974	1012	1046	1021	970	948	1014
11	1070	1099	1111	1080	1133	1150	1129	1070	1031	1127
12	1162	1193	1214	1176	1233	1262	1227	1166	1144	1223
13	1266	1314	1316	1272	1343	1358	1324	1260	1246	1301
14	1365	1416	1419	1383	1440	1455	1426	1344	1339	1413
15	1456	1513	1511	1491	1553	1549	1520	1441	1458	1508
16	1556	1601	1593	1602	1670	1659	1615	1548	1546	1610

The motivation of modern calculations of π to many decimal places was conjectured by Professor P. S. Jones [45] in 1950 as being attributable to "intellectual curiosity and the challenge of an unchecked and long untouched computation." This reason for undertaking such work should be supplemented by reference to the recurrent interest in determining a statistical measure of the randomness of distribution of the digits in the decimal representation of π.

Augustus De Morgan [46] drew attention to the deficiency in the number of appearances of the digit 7 in Shanks's 607-place approximation to π. In 1897 E. B. Escott [47] raised the question whether the deficiency of 7's noted in Shanks's final approximation could be explained.

In June 1949, the late Professor John von Neumann expressed an interest in utilizing the ENIAC to determine the value of π and e to many places as the basis for a statisitcal study of the distribution of their decimal digits. A statistical treatment of the first 2000 decimal digits of both π and e was published by N. C. Metropolis, G. Reitwiesner, and J. von Neumann [48]. Further analysis of these data was performed by R. E. Greenwood [49], using the coupon collector's test. A count of each of the decimal digits appearing in the NORC approximation appears in the paper of Nicholson and Jeenel [43]. A number of recent investigators have discussed the distribution of digits in Shanks's approximation and in the corrected value of π. These investigators include F. Bukovszky [50], W. Hope-Jones [51], E. H. Neville [52], and B. C. Brookes [53].

The writer has recently completed a count by centuries of the 16167 decimal digits constituting the fractional part of the latest approximation to π. An abridgment of this information is presented in the accompanying table.

The standard χ^2 test for goodness of fit reveals no abnormal behavior in the distribution of digits in this sample; in particular, there appears to be no basis for supposing that π is not simply normal [54] in the decimal scale of notation. It has been pointed out recently by Ivan Niven [55] that the normality of such numbers as π, e, and $\sqrt{2}$ has yet to be proved.

Numerical studies directed toward the empirical investigation of the normality of π clearly require increasingly higher decimal approximations, which can best be obtained by use of ultra-high-speed electronic computers now under design and development.

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Did You Know That π Has Been Calculated to 100,265 Decimal Places?

Daniel Shanks and John W. Wrench, Jr., both of the David Taylor Model Basin in Washington, D.C., have calculated the values of π and e to 100,265D on an IBM 7090 system.[1] The computation of π was performed July 29, 1961 at the IBM Datacenter in New York and required 8 hours 43 minutes while the evaluation of e required 2.5 hours.

Before going into the question of *why* such calculations are made, let's backtrack a bit and see what has been done in the past.

The Bible is content with a value of 3 for the ratio of the circumference of a circle to its diameter but Archimedes was able to assign limits to π between $3\frac{1}{7}$ to $3\frac{10}{71}$. The Egyptians had managed to evaluate, somehow, π as about 3.16 while the Babylonians used the same value, 3, as the Bible.

It is remarkable that the Chinese Astronomer Tsu Ch'ung-Chih discovered a simple fraction in the 5th Century that gives the value of π accurate to six decimal places:

$$\frac{355}{113} = 3.1415929 \ldots$$

From the middle of the 17th century many approximation expressions in the form of infinite series of one kind or another were developed. Evaluations of π to as many decimal places as the patience of the computer could stand followed rapidly.

π was computed to 72D by Abraham Sharp in 1699; to 127D by Fautet de Lagny in 1719; to 49D by the Japanese computer Matsunaga Ryohitsu using a power series developed by Sir Isaac Newton; to 140D in 1794 by Baron Georg von Vega (but only his first 136D were correct); to 40D by Chu Hung in 1800; to 152D by an unknown computer at the close of the 18th century; to 208D by William Rutherford in 1841, using one of Euler's arctangent relations; to 261D (twice by different methods) by W. Lehmann in 1853.*

The most celebrated calculation of π was made to 707D by William Shanks on and off for the 20-year period from 1853 to 1873. It was not until 1945 that Shanks was found, by D. F. Ferguson, to have erred at the 528th decimal place.

J. W. Wrench, Jr. and D. F. Ferguson calculated π to 808D in 1947 to match the evaluation of e at that time.

*The reverence shown to π calculators can be gained by considering that, in Germany, the value of π to 35 decimal places is called the Ludolphian number in memory of Ludoph van Ceulen, a German mathematician. Van Ceulen, in 1596, calculated π to 35D and requested that this value be inscribed on his tombstone as an epitaph. He died at the age of 70 and the tombstone was dutifully inscribed as requested.

All the above calculations were done longhand (including Shanks 707D!) or with a desk calculator. Subsequently, electronic computers were used and extended π evaluations followed: to 1120D in June 1949; to 2037D in September 1949 (taking 70 hours); to 3093D in November 1954 and January 1955 (taking only 13 minutes); to 10000D (in 1 hour 40 minutes) in January 1958 by Francois Genuys on an IBM 704 Electronic Data Processing System in Paris; and, *almost* finally, to 16167D in July 1959.

The latest calculation is that mentioned in the first paragraph.

Simon Newcomb, the astronomer and mathematician, once remarked about π that ten decimal places would suffice to give the circumference of the earth accurate to a fraction of an inch and that thirty decimal places would give the circumference of the known *universe* to microscopic accuracy!

Why in the world is such apparently pointless work being done?

One practical reason is that new computers can be checked by programming problems with known answers.

The more interesting reason—more interesting to recreational mathematicians, anyway—is to find out, by actual calculation, whether such numbers as π, e or $\sqrt{2}$ are "normal" numbers. That is, whether the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 occur in a statistically random distribution—do they each occur approximately 10% of the time?

A count of the first 16000D of π reveals no abnormal distribution. π appears to be a "normal" number.[2] At present there is no *proof* of the normality of such numbers. It is not even known if the ten consecutive digits 1234567890 occur at least once in the infinite decimal evaluation of π.

Shanks and Wrench estimate that computers will become available, in the next 5 to 7 years, which will be able to calculate π to 1,000,000 decimal places. (The IBM 7090 which performed the feat to 100,265D in 8 hours 43 minutes would require *months* to do the calculation to 1,000,000D.)

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Epilogue

We have observed that in general, two methods of thinking have been employed in computing the value of π: (1) the geometric approach, and (2) the analytical approach. It should be noted that the man who used the first of these methods thought of π as equivalent to a geometrical ratio, even as the Greek geometers considered the ratio of two line segments when studying metric properties of geometric figures. On the other hand, mathematicians using the second method think of π not as the ratio of two lengths, but as the symbol for a specific number (like the number $e = 2.718\ldots$) which enters many fields of mathematical analysis from theoretical considerations rather than from any question of practical measurement. In this connection, one of the most remarkable of mathematical relations is that which associates π and e, namely, $e^{i\pi} + 1 = 0$. The number e is itself a unique constant, being the limit of the expansion $(1 + 1/n)^n$ as n increases without limit. We know that e is not only an irrational number, but, like π, it is also a transcendental number, that is, a number which is not the root of a polynomial equation with rational coefficients. The number i is the pure imaginary unit, where $i^2 = -1$, or $i = \sqrt{-1}$. That the product of i and π, applied to e as an exponent, should yield the simple integer -1, is indeed an amazing relation.

For Further Reading and Study

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